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On the complexity of the k-customer vehicle routing problem

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Abstract

We investigate the complexity of the k-CUSTOMER VEHICLE ROUTING PROBLEM: Given an edge weighted graph, the problem requires to compute a minimum weight set of cyclic routes such that each contains a distinguished depot vertex and at most other k customer vertices, and every customer belongs to exactly one route.

Keywords: Approximation algorithms; Vehicle routing problem

1. Introduction

Let $G = (V,E)$ be an undirected graph with $V = \{0,1,\dots,n\}$. A *route* in G is a closed walk which is either a simple cycle containing vertex 0, or a 2-edge walk $(0,i,0)$ for some $i \in V$. Let $d_{i,j} \geqslant 0$ denote the *length* (or *cost*) of edge $(i,j) \in E$. In the k-CUSTOMER VEHICLE ROUTING PROBLEM (kVRP), vehicles leave the *depot* at vertex 0, and visit the other vertices of V in order to serve the customers located there. There is a restriction that a vehicle can visit at most k vertices other than the depot, however, there is no restriction on the number of vehicles used. The objective is to find a set of routes of minimum total length, such that every vertex except for the depot is visited exactly once.

We consider the complexity and approximability of kVRP as well as some special cases which we now define. We will also consider *directed* versions of these problems, where G is assumed to be a directed graph.

- *METRIC kVRP*: In this special case G is a complete graph and d is a metric, i.e., the triangle inequality is satisfied.
- *UNWEIGHTED kVRP*: In this special case $d_{i,j} = 1$ for every $(i,j) \in E$. The objective now is to minimize the number of edges in the routes, and this is equivalent to minimizing the number of vehicles (the number of edges is equal to n plus the number of vehicles).
- *kVRP*(2): In this special case G is a complete graph and the length function d has at most two distinct values.

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- *DECISION kVRP*: In this version there is no weight function, and we ask whether there exists a feasible solution, that is a solution that only uses the edges of E.

2VRP is polynomial time solvable (even in the more general case where G is a digraph) since it can be transformed to a minimum matching problem. In contrast, Haimovich and Rinnooy Kan [3] proved that kVRP, $k \geqslant 3$ is NP-hard. Haimovich et al. [4] gave a $\frac{5}{2} - \frac{3}{2k}$ approximation for METRIC kVRP.

Bazgan et al. [2] investigated the approximability of kVRP under the *differential measure*, that compares the worst-case ratio of, on the one hand, the difference between the cost of the solution generated by the algorithm and the worst cost, and on the other hand, the difference between the optimal cost and the worst cost. They designed a $\frac{1}{2}$ differential approximation for the general case, and better bounds for special cases with either bounded values of k or under the assumption that the costs satisfy the triangle inequality. They also considered the standard approximation measure and in particular proved that kVRP(2) is not $2^{p(n)}$ approximable for any $k \geqslant 5$ and a polynomial p, unless $P = NP$.

In this paper, we show that DECISION kVRP can be solved in polynomial time for $k = 3$ and 4, but not (assuming $P \neq NP$) for $k > 4$. This implies that no constant factor approximation algorithm exists for kVRP for $k > 4$. We show however, that for $k = 3$, such an algorithm, with an approximation factor of 4, is possible. The case $k = 4$ is still open, but we present a 3-approximation algorithm for 4VRP(2). Finally, we consider the directed kVRP problem, and show that in this case there are no bounded approximation algorithms for any $k \geqslant 3$.

We will use the following tools in our constructions: For a given vector $b = (b_1, \ldots, b_n)$, a *binary b-matching* is a subgraph of G in which every vertex $i \in V$ has a degree of exactly b_i. (A binary b-matching is distinguished from an *integer* b-matching where an edge can be used more than once.) A *minimum binary b-matching* is one with minimum total edge weight. A minimum binary b-matching can be solved in polynomial time. A 2-*matching* is a b-matching with $b_1 = 2$ for $i = 1, \ldots, n$.

We will use in our proofs the following NP-complete problems.

PARTITION INTO PATHS OF LENGTH k (kPP) (see, Kirkpatrick and Hell [6], Steiner [7]): Given an integer $k \geqslant 2$ and a graph $G = (V, E)$ with $|V| = (k+1)q$, is there a partition of V into q disjoint sets $V_1, \ldots, V_q$ of $k+1$ vertices each, so that each subgraph induced by V_i has a Hamiltonian (i.e., a k-edge) path?

3-DIMENSIONAL MATCHING (see, Karp [5]): Given a set $M \subseteq W \times X \times Y$, where W, X, and Y are disjoint sets having the same number q of elements, does M contain a matching, i.e., a subset $M' \subseteq M$ such that $|M'| = q$ and no two elements of M' agree in any coordinate?

2. Undirected VRP

For the graph G, denoted by S the set of vertices that are adjacent to the depot, and by $T = V \setminus (S \cup \{0\})$.

Theorem 1. DECISION 3VRP *is polynomially solvable.*

Proof. The problem has a feasible solution iff there exists a subset $E \subseteq S \times T$, such that the degree of every $t \in T$ is 2, and the degree of every $s \in S$ is 0 or 1. This is a bipartite matching problem that can be solved in polynomial time. □

Definition 2. The graph G' is constructed from the graph $G = (V, E)$ with vertex subsets $S, T \subseteq V$ such that $S \cup T = V \setminus \{0\}$ as follows: Every vertex $s \in S$ is replaced by three vertices, s_1, s_2 and s_3, and every edge $(0, s)$ such that $s \in S$ is replaced by four edges $(0, s_3)$, (s_2, s_3), (s_1, s_2) and (s_1, s_3). Every vertex $t \in T$ is replaced by four vertices, $t_1, \ldots, t_4$, and four edges, (t_1, t_2), (t_2, t_3), (t_3, t_4), and (t_1, t_3). An edge (s, t) such that $s \in S$ and $t \in T$ is replaced by the edge (s_1, t_4), and an edge (t, u) such that $t, u \in T$ is replaced by (t_1, u_1). Note that edges of E with two ends in S are not represented in G'. The construction of G' is illustrated in Fig. 1, where $S = \{a, b, c\}$ and $T = \{u, v, w\}$.

Theorem 3. DECISION 4VRP *is polynomially solvable.*

Proof. Consider the graph G' as in Definition 2. We claim that the 4VRP instance on G has a solution iff there is a subgraph of G' with degree 2 at every $v \in V \setminus \{0\}$. Checking the latter can be turned into a binary 2-matching problem by duplicating the depot sufficiently many times (once for every edge that

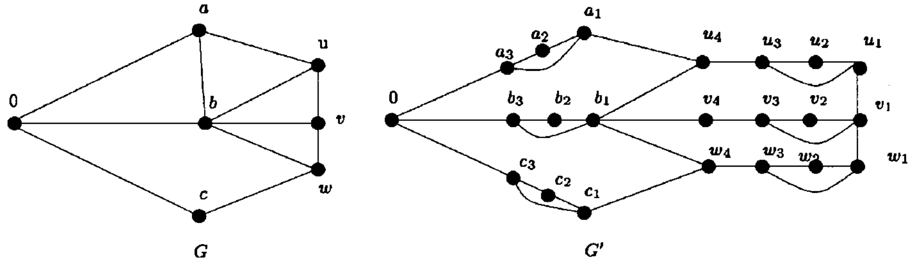

Fig. 1. The graphs G and G'.

is incident with it), and therefore it can be done in polynomial time. To prove the claim, note that the 2-matching must use (v_1, v_2) for every $v \in S \cup T$. This precludes, for example, the use of two edges of the type (u_1, v_1) and (v_1, w_1) where $u, v, w \in T$. Therefore, a cycle in the 2-matching may use at most, two consecutive vertices from T. Similarly, a cycle containing two edges of the form (u_4, b_1), (v_4, b_1), where $u, v \in T$ and $b \in S$, is not possible. Therefore, the 2-matching induces in G routes on disjoint sets of three or four customer vertices each (a route of three (four) vertices includes two vertices from S and one (two) from T, in addition to the depot). These routes cover the vertices of T. Any vertex $s \in S$ that is not used in these routes will be covered by a route $(0, s, 0)$. □

Theorem 4. Decision kVRP *is NP-complete for* $k \geqslant 5$.

Proof. We use a reduction from (kPP): Let $G(V, E)$ be an instance of kPP. We construct an instance of $(k+3)$VRP as follows: First we add a depot vertex 0. We also add a set S of $2q$ new vertices that are connected by an edge to the depot. Each vertex of S is also connected by an edge to each vertex of $V \setminus \{0\}$. It is easy to see that kPP has a solution in G iff the instance of $(k+3)$VRP has a solution (a route in the VRP instance must contain the depot, two vertices from S, and $k+1$ vertices from V). Since kPP is NP-complete for $k \geqslant 2$, it follows that the decision version of kVRP is NP-complete for $k \geqslant 5$. □

We now consider the unweighted versions of kVRP in which all the edges of G have unit length and the goal is to compute a solution of minimum total length. We first observe that we could add in the construction described in Fig. 1, an edge (a_1, b_1) for every $(a, b) \in E$, to enable routes $(0, a, b, 0)$. However, the resulting binary 2-matching still cannot account for routes of the form $(0, a, b, c, 0)$ where $a, b, c \in S$, and therefore a minimum cost binary 2-matching does not necessarily provide an optimal solution to unweighted 4VRP. Similar difficulties arise when trying to solve unweighted 3VRP, and in fact these problems are NP-hard as we now prove.

Theorem 5. unweighted kVRP *is NP-hard for* $k \geqslant 3$.

Proof. The number of edges in a solution to kVRP is at least $|V|(k+1)/k$. This value is achieved if and only if there is a solution that covers exactly k customers in every route. This, in turn, is possible if and only if the $(k-1)$PP problem in the graph induced by the customer vertices has a solution. This leads to a reduction of $(k-1)$PP to unweighted kVRP. □

Corollary 6. kVRP(2) *is NP-hard for* $k \geqslant 3$.

Remark 7. Any feasible solution to unweighted kVRP is a $2k/(k+1)$-approximation.

Proof. An optimal solution value uses at best only $(k+1)$-edge routes each covering k customers, and hence it satisfies $opt \geqslant |V|(k+1)/k$. On the other hand, the worst situation is that a solution covers the vertices by 2-edge routes, and thus its value is $sol \leqslant 2|V|$. The approximation ratio is therefore bounded by $sol/opt \leqslant 2k/(k+1)$. □

Theorem 8. *There is a polynomial* 4*-approximation algorithm for* 3*VRP*.

Proof. Consider a 3VRP instance on a graph $G = (V, E)$. Let P and P' be copies of $V \setminus \{0\}$ and let

$P'_0 = P' \cup \{0\}$. Construct a directed bipartite graph $B = (P \cup P'_0, F)$ with vertex bipartition (P, P'_0) and edge set F. F consists of all (i,j) pairs such that $i \in P$, $j \in P'_0$, and $(i,j) \in E$. Assign capacity 2 to the edges $(v,0)$ and unit capacity to all other edges in F. Assign costs to F as follows: $c_{i,j} = d_{i,j} + d_{0,j}$ for every $(i,j) \in P \times P'$, and $c_{i,0} = \frac{1}{2} d_{i,0}$ for every $i \in P$. Consider a minimum cost flow problem defined on B such that the total flow from every $v \in P$ is exactly 2 and the total flow into every $u \in P'$ is at most 1. The flow into the copy of the depot 0 in P'_0 is not restricted. This is a TRANSPORTATION PROBLEM (which can be turned into a standard minimum cost flow problem by adding a source node connected to P by edges with lower and upper bounds equal to 2, and a sink node connected to P'_0 where the edges connected to P' have unit upper bounds). An integer optimal solution $x = (x_{i,j}, (i,j) \in F)$ can be computed in polynomial time (see for example, [1]). Let $c(x) = \sum_{(i,j) \in F} c_{i,j} x_{i,j}$ be the cost of x.

We now construct a solution to the 3VRP instance of value $apx \leqslant 2c(x)$. We refer by the same index to a vertex in V and to the vertices in P and P'_0 which correspond to it. Define $R' = \{j \in P' : \sum_{i \in P} x_{i,j} = 1\}$. Let R be the subset of P with the same indices as R'.

1. For every $i \notin R$ such that $x_{i,0} = 0$, let $j,k \in R$ be the vertices such that $x_{i,j} = x_{i,k} = 1$. We construct a route $C = (0,j,i,k,0)$. The length of C is $d_{0,j} + d_{j,i} + d_{i,k} + d_{k,0} = c_{i,j} + c_{i,k}$. [Note that the sets of customers of these routes are disjoint since $i \notin R$ and the associated vertices, $j,k \in R$, are all disjoint.]
2. For every $i \notin R$ such that $x_{i,0} = 1$, let $j \in R$ be the vertex such that $x_{i,j} = 1$. We construct a route $C = (0,j,i,0)$. The length of this route is $d_{i,0} + d_{i,j} + d_{j,0} = 2c_{i,0} + c_{i,j}$. [Similarly, the sets of customers on the routes constructed in these two steps are all disjoint.]
3. For every $i \notin R$ such that $x_{i,0} = 2$, we construct a route $C = (0,i,0)$. The cost of this route is $2d_{i,0} = 4c_{i,0} = 2c_{i,0} x_{i,0}$.
4. While there exists $i \in V$ which is still not covered by any of the routes (and then $i \in R$), we construct the route $C = (0,i,0)$. Let j satisfy $x_{j,i} = 1$, then the cost of this route is $2d_{i,0} \leqslant 2c_{j,i}$.

The cost of each unit of flow has been used at most twice in the construction, so that the value of the resulting 3VRP solution satisfies $apx \leqslant 2c(x)$.

Consider an optimal solution to the 3VRP instance. We use it to construct a flow solution $\hat{x}$ such that $c(\hat{x}) \leqslant 2opt$.

1. A route $(0,i,0)$ gives $\hat{x}_{i,0} = 2$ with flow cost $2c_{i,0} = d_{i,0}$.
2. A route $(0,i,j,0)$ gives $\hat{x}_{i,0} = \hat{x}_{j,0} = 2$. The cost associated with these values is $2(c_{i,0} + c_{j,0}) = d_{i,0} + d_{j,0}$.
3. A route $(0,i,j,k,0)$ gives $\hat{x}_{i,0} = \hat{x}_{k,0} = 2$ and $\hat{x}_{j,i} = \hat{x}_{j,k} = 1$. The associated cost is $2(c_{i,0} + c_{k,0}) + c_{j,i} + c_{j,k} = (d_{i,0} + d_{k,0}) + (d_{j,i} + d_{i,0}) + (d_{j,k} + d_{k,0}) = 2(d_{i,0} + d_{k,0}) + d_{i,j} + d_{j,k}$.

We conclude that $c(\hat{x})$ is at most $2opt$, where opt is the optimal value of the 3VRP instance.

Therefore, $apx \leqslant 2c(x) \leqslant 2c(\hat{x}) \leqslant 4opt$. □

We don't know how to compute an approximation with a bounded error ratio for 4VRP, except for the following restricted case.

Theorem 9. *There is a polynomial* 3*-approximation algorithm for* $4VRP(2)$.

Proof. Suppose that the possible lengths are $q < r$. Construct from G the graph G' as in Definition 2, with the sets $S = \{v \in V : d_{0,v} = q\}$ and $T = \{v \in V : d_{0,v} = r\}$. Note that, since we assume in this problem that G is a complete graph, $S \cup T = V \setminus \{0\}$. Construct from G' a graph $\hat{G}$ by adding two parallel edges $(0, t_4)$ for every $t \in T$.

Assign lengths $\hat{d}$ to $\hat{G}$ for every $u, v \in T$ and $a \in S$ as follows:

$\hat{d}_{a_2,a_3} = \hat{d}_{a_1,a_3} = d_{0,a} = q$ (these edges correspond to a route $(0,a,0)$ in G),

$\hat{d}_{0,u_4} = d_{0,u} = r$ (these edges correspond to a route $(0,u,0)$ in G),

$\hat{d}_{a_1,u_4} = d_{a,u}$,

$\hat{d}_{u_1,v_1} = d_{u,v}$,

$\hat{d} = 0$ for the other edges of $\hat{G}$.

Compute a minimum cost binary 2-matching in $\hat{G}$ and return the corresponding routes in G, as in Theorem 3.

The construction imposes two types of constraints on the 2-matching. One is that a route cannot have more than two consecutive vertices from T. The second is that every vertex from S must be directly connected to the depot 0. [Actually, we also forbid routes $(0,a,b,0)$ with $a,b \in S$. This restriction can be avoided by adding to $\hat{G}$ the edges (a_1,b_1) with length $\hat{d}_{a_1,b_1} = d_{a,b}$. However, such a route can be replaced by routes $(0,a,0)$ and $(0,b,0)$, of total length at most twice its length.] These restrictions can be imposed without increasing the total length of the solution by more than a factor of 3. For example, if an optimal solution has a route $(0,v,s,\ldots,0)$ in which the S-vertex s is not directly connected to 0, then the approximate solution may use two routes, $(0,v,0)$ and $(0,s,\ldots,0)$ and the length is at most doubled. If a route $(0,t,t',t'',\ldots,0)$ has three consecutive vertices, $t,t',t'' \in T$, then again it can be replaced by two routes, $(0,t',t'',0)$ and $(0,t'',\ldots,0)$. The total length is at most tripled since the cost of the route is at least r, and the change adds at most $2r$. □

3. Directed VRP

We now turn to the directed version of VRP. As we will see, some of these problems are significantly harder to solve or approximate than their undirected counterparts.

Theorem 10. DIRECTED UNWEIGHTED kVRP *is NP-complete for* $k \geqslant 3$.

Proof. We describe first the proof for $k=3$. We use a reduction from 3-DIMENSIONAL MATCHING. We construct an instance of DIRECTED UNWEIGHTED 3VRP as follows $V=\{0\}\cup V_X\cup V_Y\cup V_M\cup\left(\bigcup_{w\in W}(S_w\cup S'_w)\right)$, where the vertices in V_X, V_Y, and V_M correspond to the elements of X, Y and M, respectively. $E = E_X \cup E_Y \cup E_M \cup E_{X,M} \cup E_{M,Y} \cup \left(\bigcup_{w\in W} E_w\right)$, where

$E_X = \{(0,x) : x \in V_X\}$;

$E_Y = \{(y,0) : y \in V_Y\}$;

$E_M = \{(0,m) : m \in V_M\}$;

for every $m=(w,x,y)\in M$ there is an arc $(x,m)\in E_{X,M}$ and an arc $(m,y)\in E_{M,Y}$; for every $w\in W$, let n_w be the number of times w is present in M. S_w contains n_w-1 vertices, S'_w is a copy of S_w, and for every $s\in S_w$ and its copy $s'\in S'_w$, E_w contains arcs (s,s') and $(s',0)$. In addition to these arcs, for every $s\in S_w$ and for every vertex $m\in V_M$ that contains a coordinate $w\in W$, E_w contains an arc (v,s).

Fig. 2 illustrates the construction with $q=2$ and

$$M=\{(x_1,w_1,y_1),(x_1,w_2,y_1),(x_1,w_2,y_2),\\(x_2,w_1,y_2),(x_2,w_2,y_2)\}.$$

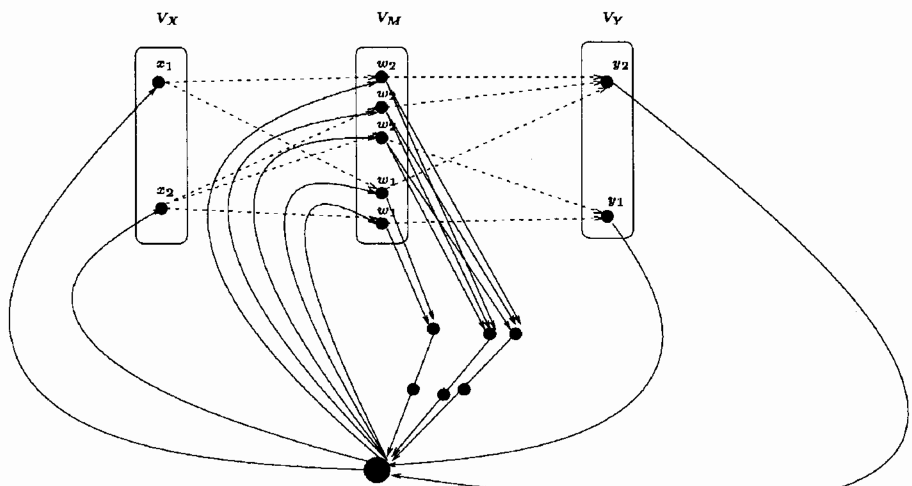

Fig. 2. Reduction from 3-dimensional matching to directed 3VRP.

A solution to 3VRP exists iff there is a 3-dimensional matching M'. There should be a route $(0, x, w, y, 0)$ for every element in M', and all the other $n_w - 1$ vertices $v \in V_M$ corresponding to w, should use routes $(0, v, s, s', 0)$ with distinct vertices $s \in S_w$.

The proof that kVRP is NP-complete for $k > 3$ is similar. The only change is that we replace every arc that leaves the depot by a sequence of $k - 2$ arcs connected in series, thus adding $k - 3$ vertices to any route. □

Corollary 11. DIRECTED kVRP *is not* $2^{p(n)}$ *approximable for any* $k \geqslant 3$ *and a polynomial* p, *unless* $P = NP$.

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